Math 220B Complex Analysis Final Exam March 16th, 2020

Name (PRINT):______________________________ PID:______________

Signature: ______________________________

Important Instructions: You may use without proof any statement in the book except when you are asked to prove it. However, you cannot use any result in your homework unless you reprove it. The points you get on all seven questions will be added. But the maximal total points you can get is 100.

Problem	Points	Score
#1	15	
#2	15	
#3	15	
#4	15	
#5	15	
#6	15	
#7	15	
Total	100	

(5) **1 (a).** State the Riemann Mapping Theorem.

(10) **1 (b).** Let $G \neq \mathbb{C}$ be a simply connected region in $\mathbb{C}$. Let $f : G \to G$ be an analytic function and assume $f(z)$ is not identically equal to z, then f has at most one fixed point G.

(15) **2.** Let f be an entire function. Assume $f^{-1}(\mathbb{Q})$ is not countably infinite. Here $\mathbb{Q}$ denotes the set of rational points on the real axis. Prove f is constant.

(15) **3.** Let $G = \{z = x + iy \in \mathbb{C} : y > x > 0\}$. Assume f is a continous function on $\overline{G}$ that is analytic in G. Here $\overline{G}$ denotes the closure of G in $\mathbb{C}$. Assume $f(z) = 0$ for every $z = x + ix$ with $1 \leq x \leq 2$. Prove $f \equiv 0$.

(15) **4.** Let $G = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Find all analytic automorphisms of G, i.e., find all one-to-one and onto analytic functions from G to G.

(15) **5.** Let $\Omega \neq \mathbb{C}$ be a simply connected region with $a \in \Omega$ and f a one-to-one analytic function from Ω onto the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Assume $f(a) = 0$ and $f'(a) > 0$. Prove that

$$\inf_{z \in \partial\Omega} |z - a| \leq \frac{1}{f'(a)} \leq \sup_{z \in \partial\Omega} |z - a|.$$

(15) **6.** Let $\{p_n\}_{n=1}^{\infty} \subset \mathbb{Z}^+$ be the strictly increasing sequence of prime numbers. Prove there exists $f \in H(\mathbb{C})$ such that for each $n \geq 1$, the Taylor expansion of f at p_n is given by

$$f(z) = p_{n+1}(z - p_n) + \text{ higher order terms in } (z - p_n)\ .$$

(15) **7.** Suppose f is an entire function, and suppose the sequence of derivatives $f', f'', f^{(3)}, f^{(4)}, \cdots$ converges uniformly on all compact sets of $\mathbb{C}$ to a limit function that is not identically zero. Prove the existence of a natural number N such that $f^{(n)}(z) \neq 0$ when $|z| < 1$ and $n > N$.